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A SURVEY OF DONALDSON THEORY

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Abstract. The purpose of this series of lectures is to provide an informal survey of the background in Donaldson theory required to understand the motivation behind Witten's construction (in 1988) of the first topological quantum field theory.

1. The Moduli Spaces

B will denote a compact, simply connected, oriented, smooth 4-manifold and we will write

$$SU(2) \hookrightarrow P_k \xrightarrow{\pi_k} B$$

for the principal $SU(2)$-bundle over B with Chern class k. $\mathcal{A}(P_k)$ is the set of all smooth (C^∞) connection 1-forms on P_k and $\mathcal{G}(P_k)$ is the group of all automorphisms of the bundle (diffeomorphisms f of P_k onto itself which satisfy $\pi_k \circ f = \pi_k$ and $f(p \cdot g) = f(p) \cdot g$ for all $p \in P_k$ and $g \in SU(2)$). $\mathcal{G}(P_k)$ is called the **gauge group**, its elements are called **(global) gauge transformations** and it acts on $\mathcal{A}(P_k)$ on the right by pullback, i. e., for any $\omega \in \mathcal{A}(P_k)$ and any $f \in \mathcal{G}(P_k)$ we have $\omega \cdot f = f^*\omega \in \mathcal{A}(P_k)$. Two connections ω and ω' are said to be **gauge equivalent** if there exists an $f \in \mathcal{G}(P_k)$ for which $\omega' = \omega \cdot f$. The set of gauge equivalence classes $[\omega]$ for $\omega \in \mathcal{A}(P_k)$ is called the **moduli space** and written $\mathcal{B}(P_k) = \mathcal{A}(P_k)/\mathcal{G}(P_k)$.

It is this moduli space that we wish to study. Unfortunately, it has no reasonable mathematical structure in the smooth context in which we have just introduced it. For this reason we must introduce appropriate Sobolev completions of the objects just described and this requires that some of the definitions be recast in other, but equivalent terms. Let us denote by $\Omega^i(P_k, su(2))$ the vector space of all i-forms on P_k with values in the Lie algebra $su(2)$. $\Omega^i_{\mathrm{ad}}(P_k, su(2))$ denotes the subspace consisting of those elements of $\Omega^i(P_k, su(2))$ that are "tensorial of