GEOMETRIC QUANTIZATION OF RIEMANN ROTORS

G. ROSENSTEEL
Department of Physics, Tulane University
New Orleans, Louisiana 70118 USA

and

E. IHRIG
Department of Mathematics, Arizona State University, Tempe, Arizona 85287 USA

ABSTRACT

Geometric quantization enables the unification of the classical and quantum theories of rotating fluids. The Riemann hydrodynamic equations of motion form a Hamiltonian dynamical system on co-adjoint orbits of a Lie subgroup GCM(3) of the noncompact Lie group Sp(3,$\mathbf{R}$). The general collective motion group GCM(3) is a 15-dimensional semidirect product of a six-dimensional abelian normal subgroup $\mathbf{R}^6$ with the motion group Gl(3,$\mathbf{R}$). The Lie algebra of $\mathbf{R}^6$ is spanned by the inertia tensor. The quantum theory of rotating fluids is constructed from irreducible unitary representations of GCM(3). The Kelvin circulation of the fluid is quantized to nonnegative integer multiples of $\hbar$.

1. Riemann's Model

Rotating systems of particles are a ubiquitous form of collectivity that is found in nature on a continuum scale spanning 35 orders of magnitude from galaxies (period T$\sim$ 200 million years $\sim 10^{15}$s), stars (T$\sim 10^6$s), and fluid droplets (T$\sim$ 1s) to atomic nuclei (T$\sim 10^{-20}$s).[1,2] A rotating system may be either classical (L$\gg \hbar$) or quantum mechanical (L$\sim \hbar$) depending upon the value of its angular momentum L compared to $\hbar$. The moment of inertia may attain the limiting cases of rigid rotation or irrotational flow or it may fall at some intermediate value.

Amidst this dynamical complexity, there is one simplifying common denominator. The shape of a rotating system is usually ellipsoidal. The size, deformation, and orientation of an ellipsoid is characterized completely by the inertia tensor,

$$Q^{\mathrm{L}}_{ij} = \sum_{\alpha} m_\alpha X_{\alpha i} X_{\alpha j}, \tag{1}$$

where m_α is the mass of particle α located at the position vector $\mathbf{X}_\alpha$ with respect to an inertial center of mass frame. For a continuous fluid, the sum is replaced by an integral over the mass density distribution. With respect to the body-fixed principal axis frame, the inertia tensor is, by definition, diagonal,

$$Q_{ij} = \sum_{\alpha} m_\alpha x_{\alpha i} x_{\alpha j} = \frac{M}{5} a_i^2 \delta_{ij}, \tag{2}$$