HALF-FORMS AND BERRY'S PHASE

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ABSTRACT

One of the more puzzling features of geometric quantization is the 'half-form' construction. In this note, I shall explain a way of looking at half-forms which is derived from recent ideas of Axelrod, Della Pietra, and Witten (1991) and Atiyah (1990), and from rather older ones of Kostant (1974). The connection between the two topics in the title is that both arise from the problem of determining how the phases of the states of a quantum system should change when external parameters are varied.

1. Holomorphic quantization

We start on familiar territory. Let (M, ω) be a symplectic manifold such that

$$\frac{1}{2\pi}\int \omega \in Z$$

for any closed 2-surface. Then there exists a Hermitian line bundle $B \to M$ with a connection with curvature ω. Let $\mathcal{H}$ be the Hilbert space of square integrable sections of B:

$$\mathcal{H} = L^2(M,B) = \{s \in \Gamma(B) \,|\, \langle s,s\rangle = \int_M (s,s)\omega^n < \infty\}.$$

For each local choice of θ such that $\omega = d\theta$, we can represent elements of $\mathcal{H}$ by complex-valued functions ψ, subject to gauge transformations

$$\psi \mapsto \psi' = e^{iu}\psi \quad \text{when} \quad \theta \mapsto \theta' = \theta + du.$$

Any $f \in C^\infty(M)$ gives rise to

(i) a Hamiltonian vector field X_f, which generates a one-parameter group of canonical transformations of M;

(ii) a symmetric operator $\hat{f}$ on $\mathcal{H}$, defined by $\hat{f}(\psi) = -i\nabla_{X_f}\psi + f\psi$;

One constructs a holomorphic quantization by picking a complex structure J that makes (M, ω) into a Kähler manifold. Then ω is given locally by $\omega = i\partial\overline{\partial}K$ for some real scalar K. The holomorphic sections of B are of the form

$$\psi = \phi e^{-K/2} \quad \text{when} \quad \theta = \tfrac{1}{2}i(\overline{\partial}K - \partial K),$$

where ϕ is holomorphic. They make up a closed subspace $F_J \subset \mathcal{H}$.