INFINITE-DIMENSIONAL SUPERMANIFOLDS OF SOLUTIONS IN LAGRANGIAN FIELD THEORIES WITH FERMION FIELDS

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Abstract

Using the supergeometric interpretation of field functionals, we show that for quite a large class of classical field models used for realistic quantum field theoretic models, an infinite-dimensional supermanifold (smf) of classical solutions in Minkowski space can be constructed. More precisely, we show that the smf of smooth Cauchy data with compact support is isomorphic with an smf of corresponding classical solutions of the model.

1. THE Φ^4 TOY MODEL: SOLUTIONS OF SOBOLEV CLASS

We start with the usual toy model of every physicist working on quantum field theory, namely the Φ^4 theory on Minkowski $\mathbb{R}^{1+3}$, given by the Lagrangian

$$L[\Phi] = \frac{1}{2}((\partial_0\Phi)^2 - \sum_{a=1}^{3}(\partial_a\Phi)^2 - m^2\Phi^2) - q\Phi^4$$

with $m, q \geq 0$. It is well-known[1] that for given Cauchy data $(\varphi^{\mathrm{Cau}}, \dot{\varphi}^{\mathrm{Cau}}) \in M_k^{\mathrm{Cau}} := H_{k+1}(\mathbb{R}^3) \oplus H_k(\mathbb{R}^3)$ (here H_k is the standard Sobolev space with order $k > 1$, in order to ensure the algebra property of H_{k+1} under pointwise multiplication) there exists a unique solution $\varphi \in M_k := C(\mathbb{R}, H_{k+1}(\mathbb{R}^3)) \subseteq C(\mathbb{R}^4)$ of the Cauchy problem

$$\frac{\delta L}{\delta\Phi}[\varphi] \equiv \Box\varphi - m^2\varphi - 4q\varphi^3 = 0, \quad \varphi(0) = \varphi^{\mathrm{Cau}}, \quad \partial_0\varphi(0) = \dot{\varphi}^{\mathrm{Cau}},$$

and that the arising nonlinear map $\Phi^{\mathrm{sol}} : M_k^{\mathrm{Cau}} \to M_k, \quad (\varphi^{\mathrm{Cau}}, \dot{\varphi}^{\mathrm{Cau}}) \mapsto \varphi$, is continuous.

As a special case of the general results described below, it turns out that this map is in fact real-analytic, and that its image, denoted M_k^{sol}, is a submanifold of the Frechet manifold M_k. Its Taylor expansion at zero arises as the solution of the "formal Cauchy problem" to find a formal power series[4] $\Phi^{\mathrm{sol}}[\Phi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}}] \in \mathcal{P}(M_k^{\mathrm{Cau}}; M_k)$ with

$$\frac{\delta L}{\delta\Phi}[\Phi^{\mathrm{sol}}] = 0, \quad \Phi^{\mathrm{sol}}(0) = \Phi^{\mathrm{Cau}}, \quad \partial_0\Phi^{\mathrm{sol}}(0) = \dot{\Phi}^{\mathrm{Cau}}$$

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